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A SIMPLE PROBLEM ON FREQUENCY OF REPETITION OF INTEGERS

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The purpose of this note is to solve the following problem :

PROBLEM : *Let X be any positive number of n digits.*

Let $1 \xrightarrow{0} X$ denote the number of zeros occurred while writing the numbers from 1 to X (Ex : $1 \xrightarrow{0} 108 = 19$). Let a_i $(i=1, 2, \ldots n)$ be the digits from right to left in the n digital number X (Ex : when $X=87576$, $n=5$ and $a_1=6$, $a_2=7$, $\ldots a_5=8$). Let $\bar{a}_i = a_i \times 10^{i-1}$ (Ex: When $X=87576$, $\bar{a}_1 = a_1 \times 10^{1-1}=6$, $\bar{a}_2=a_2 \times 10^{2-1}=70 \ldots \bar{a}_5=a_5 \times 10^{5-1}=80000$).

Then, find the value of $1 \xrightarrow{0} X$. Also find the values of $1 \xrightarrow{i} X$ $(i=1, 2, \ldots 9)$.

Solution : Let X be sufficiently large. Also suppose none of its digits is a zero. Then,

$$\left.\begin{aligned} 1 \xrightarrow{0} X = [1 \xrightarrow{0} \bar{a}_n] &+ [\{(\bar{a}_n+1) \xrightarrow{0} (\bar{a}_n+\bar{a}_{n-1})\} \\ &+ \{(\bar{a}_n+\bar{a}_{n-1}+1) \xrightarrow{0} (\bar{a}_n+\bar{a}_{n-1}+\bar{a}_{n-2})\} \ldots \\ &+ \{(\bar{a}_n+\bar{a}_{n-1}+\ldots+\bar{a}_3+1) \xrightarrow{0} (\bar{a}_n+\bar{a}_{n-1}+\ldots+\bar{a}_2)\} + \{0\}] \end{aligned}\right] \quad \ldots(1)$$

Let $1 \xrightarrow[i]{0} X$ $(i=1, 2, \ldots n-1)$ denote number of zeros occurring in the ith place from the right while writing numbers from 1 to X.

Ex : (i) $1 \xrightarrow[1]{0} 1000 = 100$ [Since 10, 20,...100, 110,...1000 are 100 in number].

(ii) $1 \xrightarrow[2]{0} 1000 = 91$ [Since 100, 101, ...109, 200, 909, 1000 are 91 in number].

Then $1 \xrightarrow{0} \bar{a}_n = [1 \xrightarrow[1]{0} \bar{a}_n] + [1 \xrightarrow[2]{0} \bar{a}_n] + \ldots + [1 \xrightarrow[n-1]{0} \bar{a}_n]$

$$= \left[\frac{\bar{a}_n}{10}\right] + \left[\frac{\bar{a}_n}{10} - 9\right] + \left[\frac{\bar{a}_n}{10} - 99\right] + \ldots + \left[\frac{\bar{a}_n}{10} - \left\{99 \ldots (n-2) \text{ times}\right\}\right]$$

$$=\left[\frac{\bar{a}_n}{10}\right]+\left[\left(\frac{\bar{a}_n}{10}+1\right)-10\right]$$
$$+\left[\left(\frac{\bar{a}_n}{10}+1\right)-10^2\right]+\ldots+\left[\left(\frac{\bar{a}_n}{10}+1\right)-10^{n-2}\right]$$
$$=\frac{(n-1)\,\bar{a}_n}{10}+(n-2)-10\left(\frac{1-10^{n-2}}{1-10}\right)$$
$$=\frac{9(n-1)\bar{a}_n+90n-(10^n+80)}{90}$$

$$(\bar{a}_n+1)\xrightarrow{0}(\bar{a}_n+\bar{a}_{n-1})=\{(\bar{a}_n+1)\xrightarrow[1]{0}(\bar{a}_n+\bar{a}_{n-1})\}+\ldots$$
$$+\{(\bar{a}_n+1)\xrightarrow[n-1]{0}(\bar{a}_n+\bar{a}_{n-1})\}$$

$$=\left\{\frac{\bar{a}_{n-1}}{10}\right\}+\left\{\frac{\bar{a}_{n-1}}{10}\right\}+\ldots+\left\{\frac{\bar{a}_{n-1}}{10}\right\}+\{10^{n-2}-1\}$$
$$=\frac{(n-2)\,\bar{a}_{n-1}+10^{n-1}-10}{10}.$$

Similarly for every other term in the equation (1) we get similar corresponding expressions.

Hence, it follows from the above inferences that

$$1\xrightarrow{0}X=\left[\frac{9(n-1)\,\bar{a}_n+90n-(10^n+80)}{90}\right]$$
$$+\left[\left\{\frac{(n-2)\,\bar{a}_{n-1}+10^{n-1}-10}{10}\right\}+\left\{\frac{(n-3)\bar{a}_{n-2}+10^{n-2}-10}{10}\right\}+\ldots+\{0\}\right] \quad \ldots(2)$$

Example : Find $1\xrightarrow{0}87962$

Solution : $X=87962$, $\bar{a}_n=80000$, $\bar{a}_{n-1}=7000$, $\ldots\bar{a}_1=2$ and $n=5$

$$1\xrightarrow{0}87962=1\xrightarrow{0}80000+80001\xrightarrow{0}87000+\ldots+87961\xrightarrow{0}87962$$
$$=\left[\frac{9\,(5-1)\,80000+90\times5-(10^5+80)}{90}\right]$$
$$+\left[\left\{\frac{(5-2)\,7000+10^{5-1}-10}{10}\right\}+\left\{\frac{(5-3)\,900+10^{5-2}-10}{10}\right\}+\ldots+\{0\}\right]$$

$$=34286$$

If however, X contains a zero say in the rth place, then the term

$$(\bar{a}_n+\bar{a}_{n-1}+\ldots\bar{a}_{r+1}+1)\xrightarrow{0}(\bar{a}_n+\bar{a}_{n-1}+\ldots+\bar{a}_{r+1}+\bar{a}_r)$$

in equation (1) vanishes but there appear at the same time $(\bar{a}_{r-1}+\bar{a}_{r-2}\ldots+\bar{a}_1)$ zeros, all in the rth place. Consequently, if all the terms of equation (2) are retained,

$$1 \xrightarrow{0} X = \left[\frac{9(n-1)\,\bar{a}_n + 90n - (10^n + 80)}{90} \right] + \left[\left\{ \frac{(n-2)\,\bar{a}_{n-1} + 10^{n-1} - 10}{10} \right\} + \left\{ \frac{(n-3)\,\bar{a}_{n-2} + 10^{n-2} - 10}{10} \right\} + \ldots + \{0\} \right] + [-10^{r-1} + (\bar{a}_{r-1} + \bar{a}_{r-2} + \ldots + \bar{a}_1) + 1] \quad \ldots (3)$$

Ex : $1 \xrightarrow{0} 57095 = 22109$ {as found from equation (2)}+ $[-10^{3-1} + (90+5) + 1]$ $= 22105_z$

By almost similar methods we can find the values of

$$1 \xrightarrow{i} X \; (i = 1, 2, \ldots 9).$$

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