K-Matrix Formalism and the Quasi-Two-Body Phase Space Factor in the Isobar Model.

R. BHANDARI

Department of Physics, New Mexico State University - Las Cruces, New Mexico 88003

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Summary. – We discuss the generalization of the results of the coupled-channel K-matrix formalism as applied to scattering of stable particles to the case which involves isobars. In the process, we develop the phase-space integral for the quasi-two-body channel and show that its upper limit is ∞ in agreement with the general principles of analytic continuation of a partial-wave amplitude. In the past, the validity of this upper limit for the integral has not been fully clear.

In extending the results of the coupled-channel K-matrix formalism for the two-body scattering to the case in which one of the inelastic channels is replaced by a quasi-two-body channel, confusion has arisen in the past (1) as to whether the upper limit of the ensuing phase-space integral for this quasi-two-body channel should be ∞ or not. A channel consisting of a stable particle and an isobar (such as the Δ, ρ, etc.) is referred to as a quasi-two-body channel. In this letter, we present a simple mathematical generalization of the results (2) for the stable two-body scattering to the case involving an isobar. In the process, we generate the afore-mentioned integral, showing that the upper limit of ∞ is a requirement imposed upon us by the general principles of analytic continuation of a partial-wave scattering amplitude into the complex-energy plane.

It is well known (3) that the S-matrix for $n+1$ coupled channels can be expressed as

$$S = \frac{1 + i\varrho^{\frac{1}{2}} K \varrho^{\frac{1}{2}}}{1 - i\varrho^{\frac{1}{2}} K \varrho^{\frac{1}{2}}}, \tag{1}$$

(1) R. S. LONGACRE, T. LASINSKI, A. H. ROSENFELD, G. SMADJA, R. J. CASHMORE and D. W. G. S. LEITH: *Phys. Rev. D*, **17**, 1795 (1978); R. A. ARNDT: private communication.

(2) See M. NAUENBERG and A. PAIS: *Phys. Rev.*, **126**, 360 (1962) and J. S. BALL, W. R. FRAZER and M. NAUENBERG: *Phys. Rev.*, **128**, 478 (1962) for past work.

(3) R. H. DALITZ and S. TUAN: *Ann. Phys.* (*N. Y.*), **10**, 307 (1960); R. H. DALITZ: *Rev. Mod. Phys.*, **33**, 471 (1961).

where the reduced K-matrix and the phase space matrix ϱ are $(n+1)\times(n+1)$ matrices. The latter is diagonal in the phase-space factors of the channels (labelled in ascending order of threshold energy) with $\varrho_{11}=\varrho_e$, the phase-space factor for the elastic channel. Since these phase-space factors are characterized in general by a threshold dependence of the form q^{2l+1}, where q is the centre-of-mass momentum and l the relative orbital angular momentum, we assume for simplicity that these are of the form

$$\varrho_{ii}=q_i^{2l_i+1}f_i(E)\,, \tag{2}$$

where E is the centre-of-mass energy and i denotes the i-th channel. The function $f_i(E)$ is a slowly varying function of E near the corresponding threshold. The unitarity condition: $S^+S=1$ implies that K is Hermitian. But time reversal renders it symmetric and thus real on the real energy axis. Consequently, K is free of threshold cuts. Writing the S-matrix further as

$$S=1+2i\varrho^{\frac{1}{2}}T\varrho^{\frac{1}{2}}\,, \tag{3}$$

where T is the reduced scattering amplitude matrix, and comparing it with eq. (1), we find

$$T=K(I-i\varrho K)^{-1}\,. \tag{4}$$

This equation yields for the reduced elastic-scattering amplitude the expression [3]

$$T_e=T_{11}=\frac{\bar{K}_e}{1-i\varrho_e\bar{K}_e}\,, \tag{5a}$$

where

$$\bar{K}_e=K_e+iK^{(0)}(I-i\varrho^{(i)}K^{(i)})^{-1}\varrho^{(i)}K^{(0)\mathrm{T}} \tag{5b}$$

and

$$T^{(0)}=(1+i\varrho_eT_e)K^{(0)}(I-i\varrho^iK^{(i)})^{-1}\,. \tag{6}$$

The conventional elastic-scattering amplitude (shown in argand plots, etc.) is of course obtained here by multiplying T_e with ϱ_e. $K_e=K_{11}$, $K^{(i)}$ is the $n\times n$ part of the K-matrix connecting inelastic channels to each other, $K^{(0)}$ is the $1\times n$ row vector that couples inelastic channels to the elastic channel, $K^{(0)\mathrm{T}}$ is its transpose, I is a $n\times n$ unit matrix, $\varrho^{(i)}$ is the $n\times n$ diagonal matrix in the inelastic phase space factors, $T^{(0)}$ is the $1\times n$ part of the first row of the T-matrix and consists of the inelastic amplitudes.

When the energy E is such that all the channels are open, ϱ_{ii} are real and the unitarity relation giving the imaginary part of elastic and inelastic amplitude reads

$$\mathrm{Im}\,T_{1j}=\sum_{k=1}^{n+1}T_{1k}(\mathrm{Re}\,(\varrho_{kk}))T_{kj} \qquad j=1,\dots,n+1\,. \tag{7}$$

The cross-sections are given by

$$\sigma_{1j}\sim|T_{1j}|^2\,\mathrm{Re}\,(\varrho_e)\,\mathrm{Re}\,(\varrho_{jj})\,, \qquad j=1,\dots,n+1\,. \tag{8}$$

If the energy is decreased to a point where only the first r channels are open, the proper procedure for analytic continuation of the amplitudes, T_{ij}, consists in letting $\varrho_{kk}\rightarrow$

$\to i|\varrho_{kk}|$ for all $k > r$. The unitarity relation, eq. (7), now contains only the first r terms for all j, and also $\sigma_{1j} = 0$ for $j > r$.

For the extension to the case involving the isobar, we first assume that there is a single inelastic channel which is quasi-two-body in nature. The production and subsequent decay of the isobar in the inelastic process is shown schematically in fig. 1. The amplitude for this process can be expressed as

$$\hat{T}(E, m) = t(E, m) B(m) , \tag{9}$$

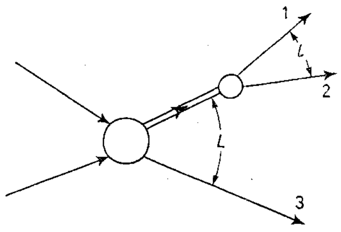

Fig. 1. – Production of an isobar in conjunction with particle 3, and its subsequent decay into particles 1 and 2. L and l are the relative orbital angular momenta in the production and decay processes, respectively.

where $t(E, m)$ is the amplitude for the production of the isobar (corresponding to a mass m) at energy E, and $B(m)$ is the amplitude corresponding to its decay. Assuming that the centrifugal barrier effects as well as any dependence of the amplitude for production of the isobar on its mass m can be factored out, we can further write

$$t(E, m) = \tilde{t}(E) k(E, m)^L f(m) \tag{10}$$

and

$$B(m) = \tilde{B}(m) q(m)^l g(m) , \tag{11}$$

where k and q are the centre-of-mass momenta in the production and decay process, respectively. The functions $f(m)$ and $g(m)$ contain any other dependence on m. They are slowly varying near $k = 0$ and $q = 0$, respectively, and do not possess any threshold behavior. The function $\tilde{B}(m)$ is the familiar Breit-Wigner propagator given by

$$\tilde{B}(m) = \frac{1}{m_0 - m - i\Gamma(m)/2} . \tag{12}$$

Now

$$\mathrm{d}\sigma(E, m) \sim |\hat{T}(E, m)|^2 k(E, m) q(m)\, \mathrm{d}m = |\tilde{t}(E)|^2\, \mathrm{d}P(E, m) , \tag{13}$$

where

$$\mathrm{d}P(E, m) = \varrho(E, m)\, \mathrm{d}m , \tag{14a}$$

$$\varrho(E, m) = k(E, m)^{2L+1} q(m)^{2l+1} |\tilde{B}(m)|^2 |F(m)|^2 , \tag{14b}$$

$$F(m) = f(m) g(m) . \tag{14c}$$

Integrating over m,

$$\sigma(E) \sim |\tilde{t}(E)|^2 P(E)\,, \tag{15}$$

where

$$P(E) = \int\limits_{M_T}^{E-M_3} \varrho(E, m)\, \mathrm{d}m\,, \tag{16}$$

$M_T = M_1 + M_2$. The integral in eq. (16) implies the sum of contributions of a continuum of two-body states with threshold for production varying from $M_T + M_3$ to E. The limit $E \to \infty$ corresponds to the situation in which all the effective two-body states, *i.e.* with threshold energies extending up to $E = \infty$, participate in the production process. In other words, the upper limit of the integral in eq. (16) becomes ∞. If E is subsequently decreased to some finite value, then, for the closed (effective) two-body channels with threshold energies in the continuum between E and ∞, the analytic continuation takes place by letting $\varrho(E, m) \to i|\varrho(E, m)|$ for all m between $E - M_3$ and ∞. Thus if we write

$$\varphi(E) = \int\limits_{M_T}^{\infty} \varrho(E, m)\, \mathrm{d}m\,, \tag{17}$$

then

$$\varphi(E) = P(E) + i\,\mathrm{Im}\,(\varphi(E))\,. \tag{18}$$

where $\mathrm{Im}\,\varphi(E)$ originates from integration over the phase space for a continuum of the closed two-body channels for which m varies from $E - M_3$ to ∞. In what follows we explicitly calculate $\tilde{t}(E)$ within the framework of the K-matrix formalism. The resulting form of $\tilde{t}(E)$ inolves not $P(E)$, but $\varphi(E)$.

Consider the following special case for $n + 1$ coupled channels when there are only stable particles present:

$$K = \begin{bmatrix} K_e & K_0 & K_0 & \dots & K_0 \\ K_0 & K_i & K_i & \dots & K_i \\ K_0 & K_i & K_i & \dots & K_i \\ \vdots & \vdots & \vdots & & \vdots \\ K_0 & K_i & K_i & \dots & K_i \end{bmatrix}, \qquad \varrho = \begin{bmatrix} \varrho_e & 0 & 0 & \dots & 0 \\ 0 & \varrho_1 & 0 & \dots & 0 \\ 0 & 0 & \varrho_2 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & \varrho_n \end{bmatrix}. \tag{19}$$

For the above special case, T_e and $T^{(0)}$, given by eqs. (5) and (6), respectively, reduce to

$$T_e = \frac{\bar{K}_e}{1 - i\varrho_e \bar{K}_e}\,, \tag{20a}$$

where

$$K_e = K_e + i\,\frac{K_0^2 \sum\limits_{j=1}^{n} \varrho_j}{1 - iK_i \sum\limits_{j=1}^{n} \varrho_j} \tag{20b}$$

and

$$T_{1k} = T_0\,, \qquad 1 < k \leqslant n+1\,, \tag{21a}$$

where

$$T_0 = (1 + i\varrho_e T_e)\frac{K_0}{1 - iK_i\sum_{j=1}^{n}\varrho_j}\,. \tag{21b}$$

If we now envision a quasi-two-body channel as a continuum of two-body states with the amplitude for production factorizable as in eq. (10), the reduced K-matrix representing the coupling of the elastic channel and this quasi--two-body channel assumes the form of eq. (19), with $n \to \infty$. The ϱ-matrix is also enlarged to a matrix of infinite dimensionality with each diagonal element in the part $\varrho^{(i)}$ equal to $\varrho(E, m)$ of eq. (14b). As a consequence, in eqs. (20b) and (21b),

$$\sum_{j=1}^{n}\varrho_j \to \int_{M_T}^{\infty}\varrho(E, m)\,\mathrm{d}m \simeq \varphi(E) \tag{22}$$

of eq. (18). T_0 is now the inelastic amplitude which is a function of E only, and is to be identified with $\tilde{t}(E)$ of eq. (10).

Thus we see that the treatment of an isobar which in principle involves matrices of infinite dimensions is reduced to one in which only a calculation of an integral is required. This integral called the quasi-two-body phase space factor plays the same role as the phase space factor in the stable two-body case. Its analytic structure, however, is very different, and is discussed in detail elsewhere (4).

If now there are two or more inelastic channels (all quasi-two-body in nature), and we consider the production of each isobar as an entirely separate reaction, the results obtained from a similar treatment such as above can be cast into the forms of eqs. (5) and (6). We refrain from showing this explicitly here because of the long, tedious (although straightforward) algebra involved. However, we do wish to emphasize that the $\varrho^{(i)}$ matrix is composed of Φ_i's, each one of which is a complex number (5). The $K^{(i)}$ matrix expresses coupling among the inelastic channels and $K^{(0)}$ between elastic and inelastic channels, as before. In fact, eqs. (5) and (6) are general equations for coupling of multi-channels regardless of the nature of the individual inelastic channel. The phase space factors in $\varrho^{(i)}$ are of the form as expressed in eq. (2) or eq. (9), according to as the channel is composed of stable particles or is a quasi-two-body channel.

Finally, if we redefine the scattering amplitude matrix as

$$T' = (\mathrm{Re}\,(\varrho))^{\frac{1}{2}}K(I - i\varrho K)^{-1}(\mathrm{Re}\,(\varrho))^{\frac{1}{2}}\,, \tag{23}$$

then

$$\sigma_{ij} \sim |T'_{ij}|^2\,. \tag{24}$$

The form in eq. (23) satisfies the unitarity relation

$$\mathrm{Im}\,(T') = T'^{+}T'\,, \tag{25-17a}$$

(4) R. BHANDARI: *Phys. Rev. D*, **25**, 1262 (1982).

(5) The Φ_i's used in (1) are real numbers.

which is the equivalent of

$$\mathrm{Im}\,(T) = T^{+}(\mathrm{Re}\,(\varrho))\,T\,, \qquad (25\text{-}17b)$$

where T is of the form as defined in eq. (4). It may also be noted that T is real analytic.

In summary, we have shown that the results of scattering involving stable particles, when generalized, yield a phase-space integral for a quasi-two-body channel. The upper limit of this integral is infinity, as a consequence of which the integral is a complex number. The need for such a clarification is important especially in view of the isobar approximation frequently used in the hadronic scattering phenomenology for the treatment of three-body final states. For example, in nucleon-nucleon (which is currently of great interest because of the possiblity of dibaryons), pion production at medium energies is assumed to originate in the Δ isobar produced first in conjunction with a nucleon. Also, as pointed out earlier, the phase-space integral in [1] is not a complex number.