THE D FUNCTION IN A PHENOMENOLOGICAL N/D MODEL

Ramesh BHANDARI
Physics Department, Virginia Polytechnic Institute and State University, Blacksburg, VA 24061, USA and Physics Department, New Mexico State University, Las Cruces, NM 88003, USA [1]

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Analytic expressions for the D function in a phenomenological N/D model are given. These expressions can be of considerable use in phenomenological studies of partial-wave scattering amplitudes pertaining to systems such as πN, KN, NN, etc. Extension to the case in which one of the two particles in a channel is an isobar is also considered.

1. Introduction. The N/D formalism has long been recognized [1] as a useful tool to study the dynamical content of partial-wave scattering amplitudes for systems such as πN, KN, NN, etc. Its practical use in general has, however, been limited by the massive amount of numerical work that is usually involved in its application. In hadronic phenomenology, the attempt to circumvent this difficulty has consisted in replacing the left-hand singularities of the amplitude (which are mainly of the branch point type) by an array of poles on the left-hand part of the energy axis. However, in spite of this approximation to the left-hand singularities, it turns out that the D-matrix exists as an integral, which more often than not, requires numerical calculation.

Motivated by a need for a tractable N/D model we present in this paper analytic expression for the aforementioned integral in a phenomenological formulation of the N/D model. We consider the general case of scattering involving two particles with relative orbital angular momentum l. Also near the end, we discuss the consequences of mass averaging which is required when one of the particles involved is an isobar.

2. Writing the S-matrix element for the single-channel case as

$$S_l(s) = 1 + 2\mathrm{i}\rho_l(s)\,A_l(s)\,, \tag{1}$$

the reduced scattering amplitude A_l in terms of the familiar functions, N_l and D_l, is

$$A_l(s) = N_l(s)/D_l(s)\,. \tag{2}$$

The subscript l denotes the relative orbital angular momentum of the particles in the two-body channel under consideration. ρ_l in eq. (1) is the corresponding two-body phase-space factor. The Mandelstam variable s equals the square of the center-of-mass energy. The product of ρ_l and A_l is the conventional elastic partial-wave amplitude. Each of the functions N_l and D_l is real-analytic with N_l possessing the left-hand dynamical cut and D_l the right-hand unitarity cut.

It is customary in a phenomenological treatment to approximate $\mathrm{Im}\,A_l$ on the left-hand cut by a series of δ-functions, i.e.,

$$\mathrm{Im}\,A_l = -\sum_{p=1}^{n} \pi\lambda^{(p)}\delta(s - s^{(p)})\,. \tag{3}$$

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This then leads [1] to

$$N_l(s)=\sum_{p=1}^{n}\frac{\lambda^{(p)}D_l(s^{(p)})}{s-s^{(p)}}\,,\qquad D_l(s)=1-\frac{1}{\pi}\sum_{p=1}^{n}\lambda^{(p)}D_l(s^{(p)})\int_{s_1}^{\infty}\frac{\rho_l(s')\,\mathrm{d}s'}{(s'-s^{(p)})(s'-s)}\,, \tag{4a,b}$$

where s_1 is the threshold energy squared, and $D_l(s)\to 1$ as $s\to\infty$. Under the approximation of eq. (3), $N_l(s)$ is represented by a series of poles simulating the effect of the left-hand cut. In phenomenological studies, the pole positions $s^{(p)}$ and the force parameters $\lambda^{(p)}$ are determined by fitting the resulting expression for the partial-wave scattering amplitude to the given set of predetermined phase-shifts. Our primary aim here is to obtain an analytic expression for $D_l(s)$ given in eq. (4b). Clearly, it is the integral therein which we must cast into an analytic form. This we do below.

For a two-body channel, it is well-known that $\rho_l(s)$ near threshold behaves as q^{2l+1} where q is the center-of-mass momentum. This behavior gives rise to a square-root unitarity cut at $s=s_1$ in the complex s-plane. Consequently, for phenomenological purposes, one may define

$$\rho_l(s)=[(s-s_1)/(s-\alpha)]^{l+1/2}\,. \tag{5}$$

The factor in the denominator of eq. (5), which incidentally gives rise to a left-hand branch point at α, is necessary to ensure the convergence of $\rho_l(s)$, i.e., as $s\to\infty$, $\rho_l(s)\to 1$. This parameter α can be fixed at a convenient value or simply set to zero. The form, eq. (5), for $\rho_l(s)$ has been effectively used before [2]. Using it in eq. (4b), one easily sees that the calculation of the D_l function involves integrals of the type

$$I_l(s)=\int_{s_1}^{\infty}\left(\frac{s'-s_1}{s'-\alpha}\right)^{l+1/2}\frac{\mathrm{d}s'}{(s'-s^{(p)})(s'-s)}\,, \tag{6}$$

whose analytic expression we are interested in.

Via a series of substitutions, we can easily cast the integral I_l into the following form:

$$I_l=(c-b)^{-1}(cI_c^{(l)}-bI_b^{(l)})\,, \tag{7}$$

where

$$I_c^{(l)}=\int_0^1\frac{(1-z)^{l+1/2}}{a-cz}\,\mathrm{d}z\,,\qquad I_b^{(l)}=\int_0^1\frac{(1-z)^{l+1/2}}{a-bz}\,\mathrm{d}z\,, \tag{8a,b}$$

and

$$a=s_1-\alpha>0\,,\qquad b=s^{(p)}-\alpha\,,\qquad c=s-\alpha\,.$$

We can further express the integral $I_c^{(l)}$ as

$$I_c^{(l)}=\frac{1}{c}\sum_{n=1}^{l}\left(\frac{c-a}{c}\right)^{n-1}J_{l-n}+\left(\frac{c-a}{c}\right)^{l}I_c^{(0)}\,, \tag{9}$$

where

$$J_m=\int_0^1(1-z)^{m+1/2}\,\mathrm{d}z=\frac{2}{2m+3}\,,\qquad I_c^{(0)}=\frac{2}{c}+\frac{(c-a)^{1/2}}{c^{3/2}}\left(\ln\frac{1-[(c-a)/c]^{1/2}}{1+[(c-a)/c]^{1/2}}+\mathrm{i}\pi\right)\,. \tag{10,11}$$

$I_b^{(l)}$ has the same series expansion as $I_c^{(l)}$ in eq. (9), except that c is replaced by b. The expression for $I_b^{(0)}$ is

$$I_b^{(0)} = \frac{2}{b} + \mathrm{i}\,\frac{(a-b)^{1/2}}{b^{3/2}}\left(\ln\frac{1-\mathrm{i}[(a-b)/b]^{1/2}}{1+\mathrm{i}[(a-b)/b]^{1/2}} + \mathrm{i}\pi\right) \tag{12}$$

($b > 0$ is assumed here and hereafter).

3. Analytic structure of $I_l(s)$. The integral $I_l(s)$ in eq. (6) is given by eqs. (7)–(12). However, to understand its general analytic properties, one need only consider the case corresponding to $l = 0$. Examination of $I_c^{(0)}$ in eq. (11) shows that a right-hand branch cut exists at $s = s_1$ (see fig. 1) with the discontinuity across it given by

$$\lim_{\epsilon\to 0}\,[I_c^{(0)}(s+\mathrm{i}\epsilon) - I_c^{(0)}(s-\mathrm{i}\epsilon)] = 2\pi[(s-s_1)/(s-\alpha)]^{1/2}(s-\alpha)^{-1}\,. \tag{13}$$

When $s < s_1$, $[(c-a)/c]^{1/2} \to \mathrm{i}[(a-c)/c]^{1/2}$ and $I_c^{(0)}$ becomes real. Since $I_b^{(0)}$ has no energy dependence and is always real, $I_0(s)$ is real-analytic.

(i) *At* $s = s^{(p)}$.

(a) *On I (principal) sheet of the* $s = s_1$ *cut.* When $s \to s^{(p)}$, $c \to b$ and $(c-a) > 0 \to (b-a) < 0$. As a consequence, $(c-a)^{1/2} \to \mathrm{i}(a-b)^{1/2}$, implying $I_c^{(0)} = I_b^{(0)}$. We have here a situation involving 0/0 for the value of I_0. Expansion of $I_c^{(0)}$ around $c = b$ shows that

$$\lim_{s\to s^{(p)}} I_0(s) = \frac{\pi a}{2b^{3/2}(a-b)^{1/2}} - \frac{\mathrm{i}a}{2b^{3/2}(a-b)^{1/2}}\ln\frac{1-\mathrm{i}[(a-b)/b]^{1/2}}{1+\mathrm{i}[(a-b)/b]^{1/2}} - \frac{1}{b}, \tag{14}$$

which is a finite quantity.

(b) *On II sheet of the* $s = s_1$ *cut.* On this sheet of the cut reached by burrowing through it from the top or the bottom (see fig. 1), $(c-a)^{1/2} \to -\mathrm{i}(a-c)^{1/2}$ for $s < s_1$ in the expression for $I_c^{(0)}$. When $c = b$, only the log terms cancel. Thus

$$I_0(s = s^{(p)})|_{\mathrm{II}} = \lim_{s\to s^{(p)}} \frac{2\pi}{(s-s^{(p)})}\left(\frac{s_1 - s^{(p)}}{s^{(p)} - \alpha}\right)^{1/2} + \text{the finite result of eq. (14)}\,, \tag{15}$$

implying there is a pole on the II sheet with a residue $= 2\pi[(s_1 - s^{(p)})/(s^{(p)} - \alpha)]^{1/2}$. This fact is also verified from consideration of pinching singularity that appears at $s = s^{(p)}$ in the analytic continuation of the integral representation of $I_0(s)$, performed by deforming the contour of integration.

As a result of the foregoing analytic structure of $I_l(s)$, $D_l(s)$ has a pole at $s = s^{(p)}$ only on the II sheet of its cut. Furthermore, since $N_l(s)$ has poles on all the sheets of the cut associated with $D_l(s)$, the amplitude $A_l(s) = N_l(s)/D_l(s)$ possesses a pole at $s = s^{(p)}$ only on the I (physical) sheet, as one expects. In a similar way, one can show that the branch point at $s' = \alpha$ in the integrand of eq. (6) is generated in complex s-plane at $s = \alpha$ but only on the unphysical sheets associated with the cut at $s = s_1$.

For completeness, we mention that near $c = a$, i.e., $s = s_1$,

$$\mathrm{Im}\, I_0(s) = [\pi/(c-b)]\,[(c-a)/c]^{1/2}\,, \tag{16a}$$

for $c - a > 0$, while

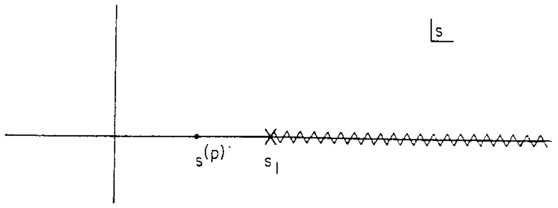

Fig. 1. The complex s-plane showing the right-hand unitarity cut at s_1 and the position $s^{(p)}$ of one of the input poles representing the function $N_l(s)$.

$$\mathrm{Real}\, I_0(s) = \frac{1}{b^{1/2}(a-b)^{1/2}}\left(\pi - \mathrm{i}\ln\frac{1-\mathrm{i}[(a-b)/b]^{1/2}}{1+\mathrm{i}[(a-b)/b]^{1/2}}\right) + \mathrm{O}((c-a)/c)\,. \tag{16b}$$

Furthermore, for the case $s \to \infty$, one notes that $c \to \infty$ and $(c-a)/c \to 1$. Consequently, the argument of the log function in eq. (11) goes to zero. But appropriate expansion shows that in this limit, $s \to \infty$,

$$\mathrm{Im}\, I_l(s) \sim 1/s\,, \qquad \mathrm{Re}\, I_l(s) \sim \ln(s)/s\,, \tag{17a,b}$$

verifying that $D_l(s) \to 1$ as $s \to \infty$.

It may be noted here that the foregoing results, although derived for a one-channel case, are also applicable to D functions which occur as elements of a D-matrix in a coupled-channel treatment. If, however, one of these D functions pertains to a quasi-two-body channel, such as $\pi\Delta$ in πN scattering, NΔ in NN scattering, etc., an average over the variable mass m of the isobar must be performed. This averaging can be carried out as follows:

$$I_l(s) = g \int_{s_1=(m_1+m_2+m_3)^2}^{\infty} \frac{1}{(s'-\alpha)^{l+1/2}(s'-s^{(p)})(s'-s)} \times\left(\int_{m_\mathrm{T}=m_1+m_2}^{\sqrt{s}-m_3} \frac{[s'-(m_3+m)^2]^{l+1/2}(m-m_\mathrm{T})^{l'+1/2}\,\mathrm{d}m}{[(m_0-m)^2+\Gamma^2/4]\,f(m)}\right)\mathrm{d}s'\,, \tag{18a}$$

$$= g \int_{m_\mathrm{T}=m_1+m_2}^{\infty} \frac{(m-m_\mathrm{T})^{l'+1/2}}{[(m_0-m)^2+\Gamma^2/4]\,f(m)} \left(\int_{(m+m_3)^2}^{\infty} \frac{[s'-(m_3+m)^2]^{l+1/2}\,\mathrm{d}s'}{(s'-\alpha)^{l+1/2}(s'-s^{(p)})(s'-s)}\right)\mathrm{d}m\,, \tag{18b}$$

where

$$g = (\Gamma/2\pi)\, f(m_0)/(m_0-m_\mathrm{T})^{l'+1/2} \tag{19}$$

and $1/f(m)$ is some form factor in the m dependence, which also ensures the convergence of the integral in the m variable [3]. It may, for example, be of the form: $f(m) = (m-\beta)^{2l+l'+1}$ where β is a phenomenological parameter less than (m_1+m_2). m_1 and m_2 are the masses of the particles into which the isobar decays, and l' is their relative orbital angular momentum. m_3 is the mass of the stable particle produced along with the isobar. This isobar corresponds to a complex mass of $m_0 - \mathrm{i}\Gamma/2$. One may note here that, in the limit $\Gamma \to 0$, the foregoing double integral reduces to the integral of eq. (6) as expected. One also observes here that the integral over s' in eq. (18b) can be replaced by analytic expressions since it is of the type given in eq. (6). But obtaining an analytic expression for the subsequent integral in the m variable in terms of simple mathematical functions does not seem to be possible. Nevertheless, despite the lack of such expressions, one can still point out some interesting analytic properties of the function $I_l(s)$ from a study of its integral representations in eqs. (18a) and (18b). From eq. (18a), we see that

$$\lim_{\epsilon\to 0}\,[I(s+\mathrm{i}\epsilon) - I(s-\mathrm{i}\epsilon)] = \frac{2\pi\mathrm{i}g}{(s-\alpha)^{l+1/2}(s-s^{(p)})} \int_{m_1+m_2}^{\sqrt{s}-m_3} \frac{[s-(m_3+m)^2]^{l+1/2}(m-m_\mathrm{T})^{l'+1/2}\,\mathrm{d}m}{[(m_0-m)^2+\Gamma^2/4]\,f(m)} \tag{20a}$$

$$\sim (\sqrt{s}-\sqrt{s_1})^{l+l'+2} \quad \text{near } s = s_1\,. \tag{20b}$$

where $s_1 = (m_1+m_2+m_3)^2$ [3]. The integral nature of the exponent implies that the right-hand unitarity cut at $s = s_1$ in fig. 1 is no longer a square-root cut. Rather, it is of the logarithmic type with an infinite number of sheets, as discussed in ref. [3]. Eq. (20a) further suggests that the analytic continuation of $I_l(s)$ to the unphysical sheet (say II sheets) reached by burrowing from the top of the cut at $s = s_1$ can be achieved in the following way:

$$I_l(s)|_{\text{II}} = I_l(s) + \text{right-hand-side of eq. (20a)}\,, \tag{21}$$

In the lower-half of this unphysical sheet (on which the resonance poles of a scattering amplitude can exist) is present a square-root branch cut at $s = s_c = (m_0 + m_3 - \mathrm{i}\Gamma/2)^2$. This square-root cut appears due to the pole at $m_0 - \mathrm{i}\Gamma/2$ in the integrand of the integral of eq. (20a). It can be taken to run to the right and parallel to the Re s axis in fig. 1. By the procedure of analytic continuation described in detail in ref. [3], one can show that the discontinuity across this cut (i.e., for Re $s >$ Re s_c) is given by

$$\lim_{\epsilon\to 0}\,[I_l(\mathrm{Re}\, s + \mathrm{i}\,\mathrm{Im}\, s_c + \mathrm{i}\epsilon)|_{\text{II}} - I_l(\mathrm{Re}\, s + \mathrm{i}\,\mathrm{Im}\, s_c - \mathrm{i}\epsilon)|_{\text{II}}]$$

$$= -2\,\pi \mathrm{i} \times \text{residue of the pole at } m_0 - \mathrm{i}\Gamma/2 \text{ in the integrand of eq. (20a)}$$

$$\times \text{ the factors multiplying the same integral} \tag{22a}$$

$$= \frac{4\pi^2 \mathrm{i}g}{(s-\alpha)^{l+1/2}(s - s^{(p)})}\,\frac{[s - (m_3 + m_0 - \mathrm{i}\Gamma/2)^2]^{l+1/2}(m_0 - m_1 - m_2 - \mathrm{i}\Gamma/2)^{l'+1/2}}{\Gamma f(m_0 - \mathrm{i}\Gamma/2)}\,. \tag{22b}$$

In a manner analogous to the analytic continuation of $I_l(s)$ into the II sheet associated with the cut at $s = s_1 = (m_1 + m_2 + m_3)^2$ [see eqs. (20a) and (21)], the knowledge of the above discontinuity provides analytic continuation of $I_l(s)|_{\text{II}}$ into the unphysical sheet associated with the above square-root cut at $s = s_c$. Analytic continuation into this region of the complex s-plane is necessary to uncover any influential nearby poles of the scattering T-matrix that may be present.

In conclusion, we have presented analytic expressions for the integral $I_l(s)$ in a phenomenological N/D formulation for two-body scattering states. When one of the two particles is an isobar, an average over the variable mass of the isobar is required. This averaging leads to different analytic properties of the function in $I_l(s)$. Although its sheet structure is more complicated in this case, one finds one can still provide a prescription for analytic continuation of $I_l(s)$ into the unphysical regions of interest.

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