LETTERE AL NUOVO CIMENTO VOL. 38, N. 9 29 Ottobre 1983

Quarkonium Spectra with the Linear Plus Coulombic Potential in the Bethe-Salpeter Equation.

M. ARAFAH (*), R. BHANDARI and B. RAM

Physics Department, New Mexico State University - Las Cruces, NM 88003

(ricevuto il 22 Agosto 1983)

PACS. 12.40. – Models of strong interactions.

Summary. – The Bethe-Salpeter equation obtained within the framework of ladder and instantaneous approximations is solved to obtain mass spectra of various $q\bar{q}$ systems with the potential $V(r) = -\frac{4}{3}(\alpha_s/r - \lambda r - c/m_q m_{\bar{q}})$.

The charmonium and upsilon spectra have been studied extensively in the near past within the framework of nonrelativistic quantum mechanics ([1-5]). Some of these studies have included the lighter mesons such as the π-meson, ρ-meson, etc., but it has been generally felt that these lighter systems should rather be studied with relativistic equations. Consequently, equations such as the Klein-Gordon and Dirac have been used along with their variants ([6-9]). In these investigations, different authors have used different prescriptions to construct the two-body equation for the descrip-

---

(*) Present address: Physics Department, King Abdulaziz University, Saudi Arabia.

(1) E. EICHTEN, K. GOTTFRIED, T. KINOSHITA, J. KOGUT, K. O. LANE and Y. M. YAN: *Phys. Rev. Lett.*, **34**, 369 (1975); A. DE RUJULA, M. GEORGI and S. L. GLASHOW: *Phys. Rev. D*, **12**, 147 (1975); X. Y. PHAM and J. M. RICHARD: *Phys. Lett. B*, **70**, 370 (1977); N. ISGUR and G. KARL: *Phys. Rev. D*, **20**, 1191 (1979); N. BARIK and S. N. JENA: *Phys. Rev. D*, **24**, 680 (1981).

(2) R. K. BHADURI, L. E. COHLER and Y. NOGAMI: *Phys. Rev. Lett.*, **44**, 1369 (1980).

(3) R. K. BHADURI, L. E. COHLER and Y. NOGAMI: *Nuovo Cimento A*, **65**, 376 (1981).

(4) H. F. DE CARVALHO and A. B. D'OLIVEIRA: *Lett. Nuovo Cimento*, **33**, 572 (1982).

(5) C. QUIGG, H. B. THACKER and J. L. ROSNER: *Phys. Rev. D*, **21**, 234 (1980).

(6) B. RAM and R. HALASA: *Phys. Rev. D*, **19**, 3467 (1979); B. RAM: *Lett. Nuovo Cimento*, **23**, 321 (1978); **28**, 476 (1980); B. RAM and M. ARAFAH: *Lett. Nuovo Cimento*, **30**, 5 (1981); B. RAM and R. LEON: *Lett. Nuovo Cimento*, **36**, 205 (1983).

(7) J. S. KANG and H. J. SCHNITZER: *Phys. Rev. D*, **12**, 841 (1975); T. GOLDMAN and S. YANKIELOWICZ: *Phys. Rev. D*, **12**, 2910 (1975); J. F. GUNION and L. F. LI: *Phys. Rev. D*, **12**, 3583 (1975).

(8) C. L. CRITCHFIELD: *Phys. Rev. D*, **12**, 923 (1975); D. W. REIN: *Nuovo Cimento A*, **38**, 19 (1977).

(9) M. SUGAWARA: *Phys. Rev. D*, **24**, 1920 (1981); D. B. LICHTENBERG, W. NAMGUNG and J. G. WILLS: *Phys. Lett. B*, **113**, 267 (1982).

tion of the $q\bar{q}$ system. The use (10,11) of the Bethe-Salpeter (BS) equation (12), which is a true two-body relativistic equation based as it is on the only successful relativistic quantum theory, has been limited, mainly because of the problems associated with the establishment of a proper kernel and the interpretation of solutions corresponding to excitation in relative time. MITRA and collaborators (11) solve the BS equation obtained within the ladder and instantaneous approximations (12) using a Coulomb plus harmonic confining potential to obtain meson and baryon spectra. In their approach the Coulombic part is treated as a perturbation.

In the present paper we solve the BS equation within the above-mentioned approximations using the QCD motivated potential—Coulomb plus *linear*—to obtain meson masses. We do not treat the Coulombic part perturbatively.

For a $q\bar{q}$ bound state, the BS equation in momentum space is given by (13)

$$\chi_{\mathrm{P}}(q) = -\int \mathrm{d}^4k\, S_{\mathrm{F}}'^{(1)}(p_1)\, S_{\mathrm{F}}'^{(2)}(p_2)\, \bar{G}(P, q, k)\, \chi_{\mathrm{P}}(k)\,. \tag{1}$$

Here p_1 and p_2 are the final 4-momenta, p_1' and p_2' are the initial 4-momenta, and the relative momenta k (before) and q (after) are given by

$$k = \frac{m_2 p_1' - m_1 p_2'}{m_1 + m_2}\,, \qquad q = \frac{m_2 p_1 - m_1 p_2}{m_1 + m_2}\,.$$

The total 4-momentum $P = p_1 + p_2 = p_1' + p_2'$, and $m_1 = m_{\mathrm{q}}$, $m_2 = m_{\bar{\mathrm{q}}}$. In the ladder approximation one replaces the exact fermion propagators $S_{\mathrm{F}}'^{(1)}$ and $S_{\mathrm{F}}'^{(2)}$ and the interaction function $\bar{G}$ by their lowest-order values: $S_{\mathrm{F}}'(p) \approx S_{\mathrm{F}}(p)$ = free fermion propagator, $\bar{G}(P, q, k) \approx G_0(q, k)$. By means of

$$G_0(q, k) = \frac{1}{(2\pi)^4}\, F_{12}\, \gamma_\mu^{(1)} \gamma_\mu^{(2)} \langle q | V_{12} | k \rangle\,, \tag{2}$$

where F_{12} is the color factor whose value is $-\frac{4}{3}$ for $q\bar{q}$, eq. (1) takes the form

$$\chi_{\mathrm{P}}(q) = -\frac{F_{12}}{(2\pi)^4} \int \mathrm{d}^4k\, S_{\mathrm{F}}^{(1)}(p_1)\, S_{\mathrm{F}}^{(2)}(p_2)\, \gamma_\mu^{(1)} \gamma_\mu^{(2)} \langle q | V_{12} | k \rangle\, \chi_{\mathrm{P}}(k)\,. \tag{3}$$

Equation (3) describes a bound state of mass M in the centre-of-mass frame of a $q\bar{q}$ pair so that the total momentum $P = (0, 0, 0, iM)$ satisfies the relation

$$P^2 = -M^2\,. \tag{4}$$

---

(10) R. F. MEYER: *Nucl. Phys. B*, **71**, 226 (1974); M. BOHM and M. KRAMMER: *Nucl. Phys. B*, **120**, 113 (1977); C. ALABISO and G. SCHIERHOLZ: *Nucl. Phys. B*, **126**, 461 (1977); D. GROMES: *Z. Phys. C*, **11**, 147 (1981); W. KEUNG and I. J. MUZINICH: *Phys. Rev. D*, **27**, 1518 (1983).

(11) A. N. MITRA: *Z. Phys. C*, **8**, 25 (1981); A. N. MITRA and I. SANTHANAM: *Z. Phys. C*, **8**, 33 (1981); D. S. KULSHRESHTHA, A. N. MITRA and I. SANTHANAM: *Lett. Nuovo Cimento*, **34**, 220 (1982); A. N. MITRA and D. S. KULSHRESHTHA: *Phys. Rev. D*, **26**, 3123 (1982); D. S. KULSHRESHTHA, A. N. MITRA and I. SANTHANAM: *Phys. Rev. D*, **26**, 3131 (1982).

(12) M. LEVY: *Phys. Rev.*, **88**, 72 (1952); E. E. SALPETER and H. A. BETHE: *Phys. Rev.*, **84**, 1232 (1951); M. GELL-MANN and F. E. LOW: *Phys. Rev.*, **84**, 350 (1951).

(13) D. LURIE: *Particles and Fields* (New York, N. Y., 1968).

Using $S_F^{-1}(p) = i(m_q + i\gamma_\mu p_\mu)$ for the free fermion propagators for particle 1 and 2 and the Gordon decomposition for $\gamma_\mu^{(1)}$ and $\gamma_\mu^{(2)}$, one obtains from (3)

$$\chi_P(q) = -\frac{F_{12}}{(2\pi)^4}\int d^4k\, I(q,k)\,\chi_P(k)/[m_1^2 + (p_1)^2][m^2 + (p_2)^2], \tag{5}$$

where

$$I(q,k) = \langle q|V_{12}|k\rangle[4\eta_1\eta_2 P^2 - (q+k)^2 + \sigma_{\mu\nu}^{(1)}\sigma_{\mu\lambda}^{(2)}(q-k)_\nu(q-k)_\lambda + 2(\eta_2-\eta_1)P_\mu(q+k)_\mu - 2i\{\eta_2\sigma_{\mu\nu}^{(1)} - \eta_1\sigma_{\mu\nu}^{(2)}\}P_\mu(q-k)_\nu - 2i\{\sigma_{\mu\nu}^{(1)} + \sigma_{\mu\nu}^{(2)}\}q_\mu k_\nu] \tag{6}$$

with

$$\eta_1 = \frac{m_1}{m_1+m_2} \qquad \text{and} \qquad \eta_2 = \frac{m_2}{m_1+m_2}.$$

In the form (5), the BS equation has in it the q_0, k_0 dependence. This can be gotten rid of by using the instantaneous approximation (IA) in which one sets $k_0 = q_0$ and integrates over dq_0. After doing this one obtains for the BS equation

$$\frac{M}{2}(4m_1m_2 + 4\boldsymbol{q}^2 - 4\eta_1\eta_2M^2)\chi(\boldsymbol{q}) = -\frac{4}{3}\int\frac{d^3k}{(2\pi)^3}\chi(\boldsymbol{k})\cdot \left[-4\eta_1\eta_2M^2 - (\boldsymbol{q}+\boldsymbol{k})^2 - 2i(\sigma_{ij}^{(1)} + \sigma_{ij}^{(2)})q_iq_j + \sigma_{ij}^{(1)}\sigma_{il}^{(2)}(\boldsymbol{q}-\boldsymbol{k})_j(\boldsymbol{q}-\boldsymbol{k})_l - \frac{(m_1^2+m_2^2)M}{m_1m_2(m_1+m_2)}(\boldsymbol{q}^2-\boldsymbol{k}^2)\right]\langle\boldsymbol{q}|V_{12}|\boldsymbol{k}\rangle, \tag{7}$$

in terms of the Schrödinger type BS wave function

$$\chi(\boldsymbol{q}) = \int dq_0\,\chi_0(q).$$

In eq. (7), $\langle\boldsymbol{q}|V_{12}|\boldsymbol{k}\rangle$ is the Fourier transform of the $q\bar{q}$ interaction potential. The expression (6) for $I(\boldsymbol{q}, q_0; \boldsymbol{k}, k_0)$ can be written as

$$I(\boldsymbol{q}, q_0; \boldsymbol{k}, k_0) = C_0 + C_1q_0 + C_2q_0^2.$$

In writing eq. (7) we have considered only the first term C_0 which is independent of q_0. This is simply *assuming* q_0 to be small. Furthermore, we have omitted (14) a term $\sigma_{4i}^{(1)}\sigma_{4j}^{(2)}(q-k)_i(q-k)_j$ which gives the contribution $-\frac{1}{4}(\boldsymbol{q}^2-\boldsymbol{k}^2)^2/m_1m_2$. In the present investigation the effect of this term is taken into account by adding a phenomenological constant term C/m_1m_2 in the potential (15). Thus, for the $q\bar{q}$ interaction potential we take

$$V(r) = \frac{\alpha_s}{r} - \lambda r - \frac{C}{m_1m_2}. \tag{8}$$

(14) Detailed analysis including this term is under investigation.

(15) A pure constant term (independent of m_q) has normally been included in most nonrelativistic investigations. See, for example, ref. (2).

The Coulombic part is supposed to come from one-gluon exchange in QCD and the linear part from higher-order diagrams. Here we treat both the linear and Coulombic parts on the *same footing*. Note that with the color factor $(-\frac{4}{3})$ the potential (8) has the correct attractive form. The Fourier transform of (8) is

$$\langle \boldsymbol{q}|V_{12}|\boldsymbol{k}\rangle = +\frac{4\pi\alpha_s}{(\boldsymbol{q}-\boldsymbol{k})^2}+\frac{8\pi\lambda}{(\boldsymbol{q}-\boldsymbol{k})^4}-\frac{(2\pi)^3 C\delta(\boldsymbol{q}-\boldsymbol{k})}{m_1 m_2}\,. \tag{9}$$

After substituting (9) in (7) and taking the Fourier transform, we get the BS equation in co-ordinate space:

$$\begin{aligned}\frac{M}{2}[4m_1m_2-4\eta_1\eta_2M^2-4\nabla_r^2]\chi(\boldsymbol{r}) = &-\frac{4}{3}\lambda\left[-4\eta_1\eta_2M^2r+\frac{2}{r}+\frac{4\boldsymbol{r}\cdot\boldsymbol{\nabla}}{r}+\frac{4r^2\nabla^2}{r}-\right.\\ &\left.-\frac{4\boldsymbol{L}\cdot\boldsymbol{S}}{r}+\frac{M}{(m_1+m_2)}\frac{(m_1^2+m_2^2)}{m_1m_2}\left(\frac{2}{r}+\frac{2\boldsymbol{r}\cdot\boldsymbol{\nabla}}{r}\right)-\frac{S_{12}}{3r}-\frac{4}{3}\frac{\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2}{r}\right]\chi(\boldsymbol{r})+\\ &+\frac{4}{3}\alpha_s\left[4\eta_1\eta_2\frac{M^2}{r}+4\pi\delta^3(\boldsymbol{r})\left(1-\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2+\frac{M}{(m_1+m_2)}\frac{(m_1^2+m_2^2)}{m_1m_2}\right)-\frac{S_{12}}{r^3}-\right.\\ &\left.-\frac{4\boldsymbol{L}\cdot\boldsymbol{S}}{r^3}-\frac{4r^2\nabla^2}{r^3}+\left(\frac{2M}{(m_1+m_2)}\frac{(m_1^2+m_2^2)}{m_1m_2}+4\right)\frac{\boldsymbol{r}\cdot\boldsymbol{\Delta}}{r^3}\right]\chi(\boldsymbol{r})+C[-4\eta_1\eta_2M^2+4\nabla_r^2]\chi(\boldsymbol{r})\,.\end{aligned} \tag{10}$$

Note the natural appearance of the spin-spin, spin-orbit, tensor, and contact terms in this equation.

Our task now is to solve eq. (10) to get the meson masses. In anticipation of the unpleasantness which may be caused by the δ-function, we replace it by [2]

$$\delta(\boldsymbol{r}) = \lim_{r_0\to 0}\frac{1}{2\pi r_0^2}\left(\frac{1}{r}-\frac{1}{4r_0}\right)\exp\left[-\frac{r}{r_0}\right], \tag{11}$$

where we take $r_0 = \alpha_s/m$ in analogy with QED [16], and then solve eq. (10) numerically by the Runge-Kutta method using boundary conditions that ensure the proper behavior of $\chi(r)$ at small and large r.

The results of our calculations for the $b\bar{b}$, $c\bar{c}$, $s\bar{s}$, $u\bar{u}$ meson masses (M_{theor}) are given in column 3 of table I where they are also compared with the experimental masses (M_{exp}) given in column 4. The parameters which give a very good fit to $b\bar{b}$ states are found to be

$$\alpha_s = 0.6\,,\qquad \lambda = 0.08\ (\text{GeV})^2\,,\qquad C = -0.112\ (\text{GeV})^3 \tag{12}$$

with $m_b = 4.98$ GeV. With these parameters and a value $m_c = 1.535$ GeV, the theoretical values for the $c\bar{c}$ states are also in good agreement with the experimental ones for the triplet S-states. However, for the singlet $c\bar{c}$ state η_c, the theoretical value (3.036 GeV) is 56 MeV too high. Furthermore, M_{theor} for the P and D states are consistently lower than their experimental values. For the $s\bar{s}$ mesons, the triplet ground state is obtained at 1.015 GeV with $m_s = 0.44$ GeV. But with the same m_s, the second radial excitation is too low. For the lighter $u\bar{u}$ states $1\,^3S_1$ and $2\,^3S_1$, the theoretical

([16]) S. M. BLINDER: *Phys. Rev. A*, **18**, 853 (1978).

TABLE I. – *Masses in* GeV *of* $b\bar{b}$, $c\bar{c}$, $s\bar{s}$, *and* $u\bar{u}$ *states with parameters* $\alpha_s = 0.6$, $\lambda = 0.08\ (\text{GeV})^2$, $C = -0.112\ (\text{GeV})^3$, $m_b = 4.98$ GeV, $m_c = 1.535$ GeV, $m_s = 0.44$ GeV, *and* $m_u = 0.3$ GeV.

$q\bar{q}$	state $= n^{2s+1}L_J$	M_{theor}	M_{exp}
$b\bar{b}$	$1\,^3S_1$	9.465	9.460
	$2\,^3S_1$	10.044	10.020
	$3\,^3S_1$	10.342	10.350
	$4\,^3S_1$	10.569	10.570
$c\bar{c}$	$1\,^3S_1$	3.107	3.097 (J/ψ)
	$2\,^3S_1$	3.558	3.685
	$3\,^3S_1$	3.908	4.030
	$4\,^3S_1$	4.212	4.415
	$1\,^1S_0\ (\eta_c)$	3.036	2.980
	$1\,^3P_0$	2.758	3.415
	$1\,^3P_1$	3.315	3.510
	$1\,^3P_2$	3.410	3.556
	$1\,^3D_1$	3.520	3.77
	$2\,^3D_1$	3.818	4.16
$s\bar{s}$	$1\,^3S_1$	1.015	1.020 (φ)
	$2\,^3S_1$	1.310	1.680 (φ′)
	$1\,^3P_0$	0.909	
	$1\,^3P_1$	1.006	
	$1\,^3P_2$	1.093	
$u\bar{u}$	$1\,^3S_1$	0.866	0.770
	$2\,^3S_1$	1.137	1.250
	$1\,^1S_0$	0.832	0.140 (π)

TABLE II. – *Masses in* GeV *of* $c\bar{u}$, $c\bar{s}$, *and* $s\bar{u}$ *and* $b\bar{u}$ *states with parameters given in table I.*

$q\bar{q}$	state $= n^{2s+1}L_J$	M_{theor}	M_{exp}
$c\bar{u}$	$1\,^3S_1$	2.166	2.010 (D*)
	$1\,^1S_0$	2.809	1.865 (D)
$c\bar{s}$	$1\,^3S_1$	2.217	2.140 (F*)
	$1\,^1S_0$	2.133	2.020 (F)
$s\bar{u}$	$1\,^3S_1$	0.939	0.892 (K*)
	$1\,^1S_0$	0.983	0.490 (K)
$b\bar{u}$	$1\,^3S_1$	5.798	5.3 (B*)
	$1\,^1S_0$	5.767	5.3 (B)

values are reasonably close to the experimental ones with $m_u = 0.3$ GeV. But the π-meson does not match at all.

It is worth mentioning here that it was possible to obtain a reasonably good fit for the $b\bar{b}$ and $c\bar{c}$ S-states without the constant C/m_1m_2 term, say for example with $\alpha_s = 0.5$, $\lambda = 0.1\ (\text{GeV})^2$ and $m_b = 4.8$ GeV, $m_c = 1.4$ GeV; but the theoretical values 1.683 GeV and 1.885 GeV which resulted for the $1\,^3S_1$ and $2\,^3S_1$ $u\bar{u}$ states, respectively, were too high compared to their experimental counterparts 0.770 GeV and 1.250 GeV. Thus it was necessary to include the term C/m_1m_2 in the $q\bar{q}$ interaction. Note that the numerical value of C turns out to be negative (17).

The results of our calculations, which also constitute the predictions of this model, for the $c\bar{u}$, $c\bar{s}$, $s\bar{u}$, $b\bar{u}$ S-states in which the quark and the antiquark do not have the same mass are given in column 3 of table II, where they are also compared with the corresponding experimental values which are given in column 4. Except for the singlet $c\bar{u}$ and $s\bar{u}$, the theoretical results compare well with the experimental ones. This is indeed encouraging for the model used in this investigation.

* * *

The authors thank Mr. V. KRISS for helpful discussions.

(17) In this connection, see also D. GROMES: *Z. Phys. C*, **11**, 147 (1981).